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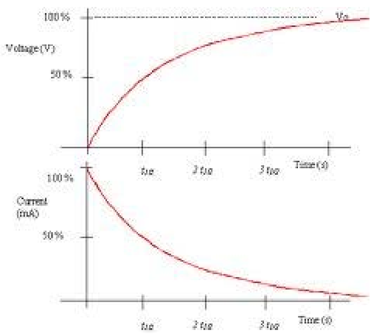
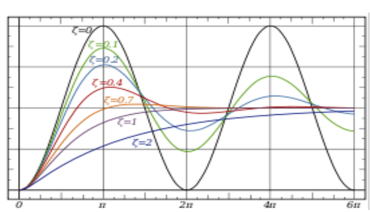
AC circuits Transient Analysis

❖ *1st order networks (RC & RL)*



❖ *2d order networks (RLC)*



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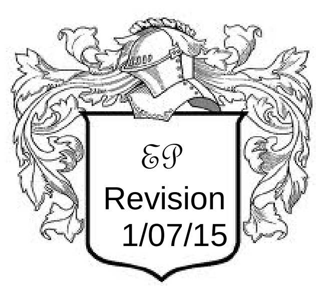


CONTENTS

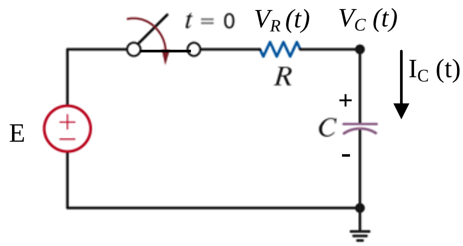
1 First Order RC Circuit Transient Analysis

Circuits containing only a single storage element are defined as first-order networks and result in a first-order differential equation (i.e. RC & RL circuits).

1.1 RC Circuit Capacitor Charging Phase

- Capacitor current I_C (t) with initial condition $V_C(0^-) = 0$

The RC Circuit analysis provides a 1st order Differential Equation when performing the transient analysis:



@t = 0, the switch is closed providing a current path in the circuit. Using Kirchhoff's voltage law:

Eq1: $E = V_R(t) + V_C(t)$

The voltage across a resistor, the voltage across a capacitor and the current through a capacitor is defined as:

Eq2: $V_R(t) = I_C(t)R$

Eq3: $V_C(t) = \frac{1}{C}\int_{t=-\infty}^{t} I_C(\tau)d\tau$

Eq4: $I_C(t) = C\frac{dV_C(t)}{dt}$

We solve for *$I_C(t)$* and thus, substitute Eq2 and Eq3 into Eq1 and divide by R on both sides

$$E = I_C(t)R + \frac{1}{C}\int I_C(t)dt$$

Eq5: $\frac{E}{R} = I_C(t) + \frac{1}{RC}\int I_C(t)dt$

Differentiate both sides of Eq5:

Eq6: $\frac{d}{dt}\left(\frac{E}{R}\right)=\frac{d}{dt}\left(I_C(t)+\frac{1}{RC}\int I_C(t)dt\right)$

Rearrange and evaluate the derivatives of Eq6, giving a first-order linear differential equation:

$$\frac{d}{dt}I_C(t)+\frac{1}{RC}\frac{d}{dt}\left(\int I_C(t)dt\right)=\frac{d}{dt}\left(\frac{E}{R}\right)$$

Eq7: $\frac{d}{dt}I_C(t)+\frac{I_C(t)}{RC}=0$

We solve differential equation Eq7 by first rearranging the equation:

$$\frac{d}{dt}I_C(t)=-\frac{I_C(t)}{RC}$$

Eq8: $\frac{d\,I_C(t)}{I_C(t)}=-\frac{1}{RC}dt$

Integrate both sides of Eq8:

$$\int\frac{d\,I_C(t)}{I_C(t)}=-\frac{1}{RC}\int dt$$

$$\ln I_C(t)+K1=\frac{-t}{RC}+K2$$

Combine the constants of integration such that K3 = K2 – K1:

Eq9: $\ln I_C(t)=\frac{-t}{RC}+K3$

Solve Eq9 for $I_C(t)$:

$$e^{\left(\ln I_C(t)\right)}=e^{\left(\frac{-t}{RC}+K3\right)}=e^{\frac{-t}{RC}}\,e^{K3}$$

Let K4 = e^{K3}

Eq10: $I_C(t)=K4\,e^{\frac{-t}{RC}}$

We solve Eq10 for K4 using initial condition $I_C(0^+) = \frac{E}{R}$ which occurs when $V_C(0^-) = V_C(0^+) = 0$:

$$I_C(0^+) = K4\,e^0 = \frac{E}{R}$$

Giving:

Eq11: $K4 = \frac{E}{R}$

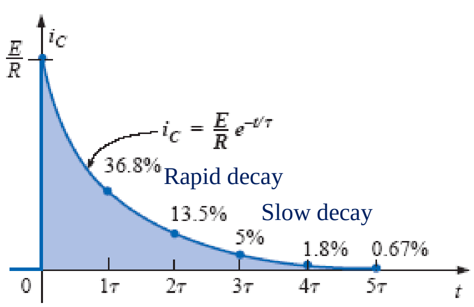
Substitute Eq11 into Eq10. We can see that when $V_C(0^-) = 0$, the current $I_C(t)$ through capacitor C is given by:

Eq12: $I_C(t) = \frac{E}{R}\,e^{\frac{-t}{RC}}$

We define RC to be the network time constant with τ =RC [sec].

We assume that steady state is reached at t = 5 τ

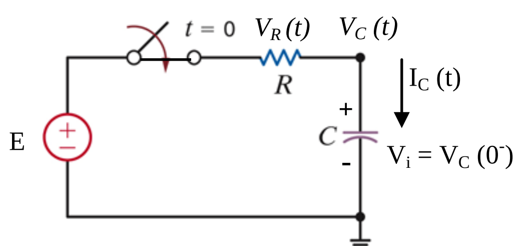
The Changes in $I_C(t)$ between time constants:



$(0 \rightarrow 1)\tau$	63.2%
$(1 \rightarrow 2)\tau$	23.3%
$(2 \rightarrow 3)\tau$	8.6%
$(3 \rightarrow 4)\tau$	3.0%
$(4 \rightarrow 5)\tau$	**1.2%**
$(5 \rightarrow 6)\tau$	0.4% ← Less than 1%

- Capacitor current I_C (t) with initial condition $V_C(0^-) = V_i$

If the initial voltage across the capacitor is not zero, we define the initial condition as $V_C(0^-) = V_i$



As a result, we get $I_C(0^+) = \frac{E - V_i}{R}$

We revisit Eq10 and solve for K4 using initial condition $I_C(0^+) = \frac{E - V_i}{R}$:

$$I_C(0^+) = K4\, e^0 = \frac{E - V_i}{R}$$

Giving:

Eq13: $K4 = \frac{E - V_i}{R}$

Substitute Eq13 into Eq10. We can see that when $V_C(0^-) = V_i$, the current $I_C(t)$ through capacitor C is given by:

Eq14: $I_C(t) = \frac{E - V_i}{R} e^{\frac{-t}{RC}}$

- Capacitor voltage V_C (t) with initial condition $V_C(0^-) = 0$

We substitute $I_C(t) = \frac{E}{R} e^{\frac{-t}{RC}}$ into $V_C(t) = \frac{1}{C} \int_{t=-\infty}^{t} I_C(\tau) d\tau$ to derive the equation for the voltage $V_C(t)$ across the capacitor when $V_C(0^-) = 0$:

$$V_C(t) = \frac{1}{C} \int I_C(t) dt = \frac{1}{C} \int \frac{E}{R} e^{\frac{-t}{RC}} dt = E \int \frac{1}{RC} e^{\frac{-t}{RC}} dt$$

We use integration by substitution with:

$$u = \frac{-t}{RC}, \; du = \frac{-1}{RC} dt$$

Giving:

$$V_C(t) = E \int \frac{1}{RC} e^{\frac{-t}{RC}} dt = -E \int e^u du = -Ee^u + K$$

Eq15: $V_C(t) = -Ee^{\frac{-t}{RC}} + K$

We solve Eq15 for K using the initial condition $V_C(0^-) = 0$:

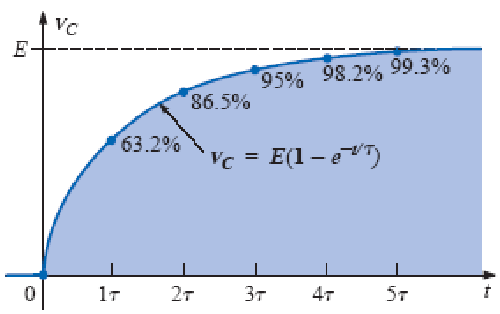
Eq16: $0 = -Ee^{0} + K \;\; \rightarrow K = E$

We substitute K into Eq15:

$$V_C(t) = E - Ee^{\frac{-t}{RC}}$$

We can see that when $V_C(0^-) = 0$, the voltage $V_C(t)$ across capacitor C is given by:

Eq17: $V_C(t) = E\left(1 - e^{\frac{-t}{RC}}\right)$

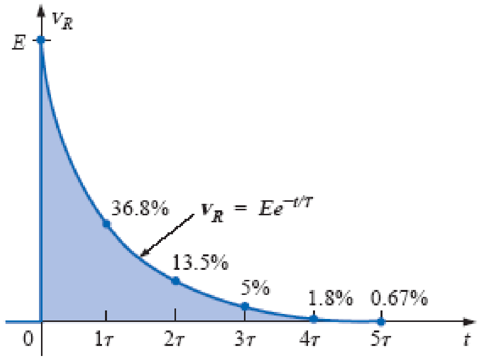


The voltage across the resistor can be found by:

$$V_R(t) = E - V_C(t)$$
$$V_R(t) = E - E(1 - e^{-t/RC})$$
$$V_R(t) = E - E + Ee^{-t/RC}$$

Thus with the initial condition $V_C(0^-) = 0$:

Eq18: $V_R(t) = E\, e^{\frac{-t}{RC}}$



- Capacitor voltage V_C (t) with initial condition $V_C(0^-) = V_i$

We substitute $I_C(t) = \frac{E - V_i}{R} e^{\frac{-t}{RC}}$ into $V_C(t) = \frac{1}{C}\int_{t=-\infty}^{t} I_C(\tau)d\tau$ to derive the equation for the voltage $V_C(t)$ across the capacitor when $V_C(0^-) = 0$:

$$V_C(t) = \frac{1}{C}\int I_C(t)dt = \frac{1}{C}\int \frac{E - V_i}{R} e^{\frac{-t}{RC}} dt = (E - V_i)\int \frac{1}{RC} e^{\frac{-t}{RC}} dt$$

We use integration by substitution with:

$$u = \frac{-t}{RC}, \ du = \frac{-1}{RC}dt$$

Giving:

$$V_C(t) = (E - V_i)\int \frac{1}{RC} e^{\frac{-t}{RC}} dt = -(E - V_i)\int e^u du = -(E - V_i)e^u + K$$

Eq19: $V_C(t) = -(E - V_i)e^{\frac{-t}{RC}} + K$

We solve Eq19 for K using the initial condition $V_C(0^-) = V_i$:

Eq20: $V_i = -(E - V_i)e^0 + K \rightarrow K = E$

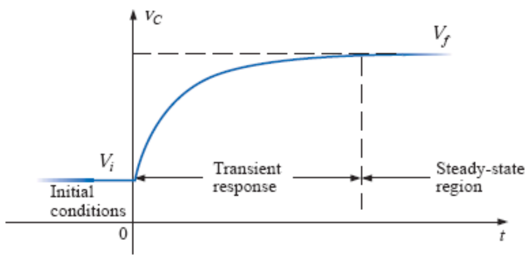
We substitute K into Eq19:

$$V_C(t) = E - (E - V_i)e^{\frac{-t}{RC}}$$

We can see that when $V_C(0^-) = V_i$, the voltage $V_C(t)$ across capacitor C is given by:

Eq21: $V_C(t) = E - (E - V_i)e^{\frac{-t}{RC}}$

One can see that Eq21 becomes Eq17 when $V_C(0^-) = 0$.



The voltage across the resistor with the initial condition $V_C(0^-) = V_i$ is found to be:

Eq22: $V_R(t) = (E - V_i)e^{\frac{-t}{RC}}$

- A Shortcut approach to find the voltage $V_C(t)$:

Substitute:

$I_C(t) = \frac{E}{R} e^{\frac{-t}{RC}}$ into $V_R(t) = I_R(t)\ R = I_C(t)\ R$

We get:

$$V_R(t) = E\ e^{\frac{-t}{RC}}$$

Given that $V_R(t) + V_C(t) = E$

We get:

$$V_C(t) = E - V_R(t) = E - E\ e^{\frac{-t}{RC}}$$

Resulting in:

$V_C(0^-) = 0 \rightarrow V_C(t) = E\left(1 - e^{\frac{-t}{RC}}\right)$

$V_C(0^-) = V_i \rightarrow V_C(t) = E - (E - V_i)e^{\frac{-t}{RC}}$

- Summary – Charging phase behavior for an RC circuit:

E represents the final voltage the capacitor reaches and $\tau = RC$ represents the circuit's time constant.

When $V_C(0^+) = 0$

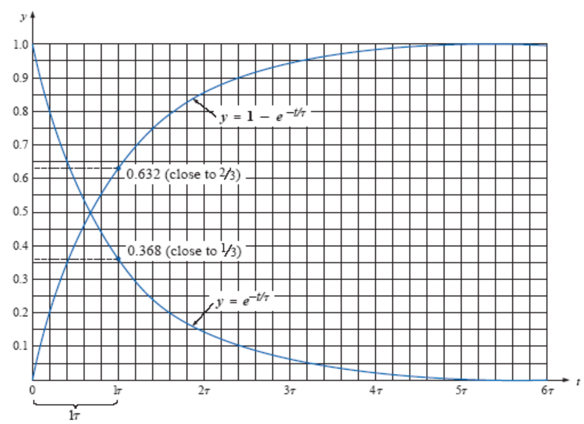
$$I_C(t) = \frac{E}{R} e^{\frac{-t}{\tau}}$$

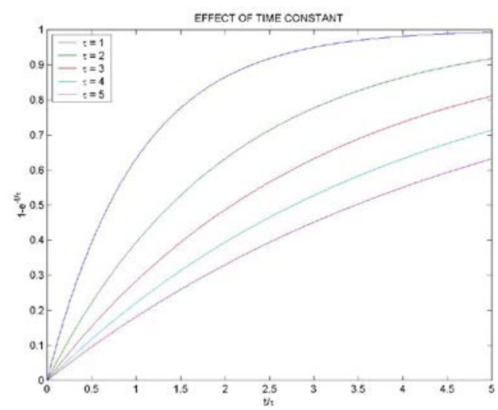
$$V_C(t) = E\left(1 - e^{\frac{-t}{\tau}}\right)$$

When $V_C(0^+) = V_i$

$$I_C(t) = \frac{E - V_i}{R} e^{\frac{-t}{\tau}}$$

$$V_C(t) = E - (E - V_i) e^{\frac{-t}{\tau}}$$





Furthermore:

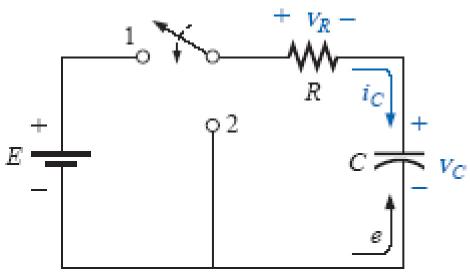
$$V_R(t) = E\, e^{\frac{-t}{RC}}$$

And

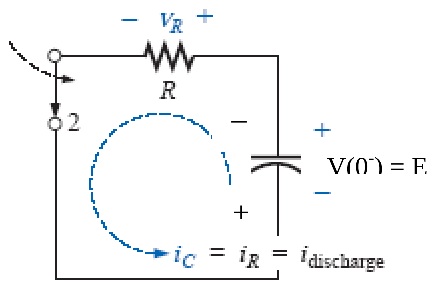
$$V_R(t) = (E - V_i)\, e^{\frac{-t}{RC}}$$

1.2 RC Circuit Capacitor Discharging Phase

We have so far analyzed the network below in its charging phase when the switch is placed in position 1.



We now throw the switch to position 2. The capacitor begins to discharge at a rate controlled by the time constant T = *RC.*



Remark:

- If the switch is moved to position 2 at after 5 T, then the capacitor would have charged fully to E. However, In general, if the switch is moved to position 2 before 5 T, then the initial capacitor voltage is V_i.
- The current reverses direction during the discharging phase

- Capacitor current I_C (t)

Using Kirchhoff's voltage law:

Eq23: $0 = V_C(t) + V_R(t)$

We get;

$$0 = I_C(t)R + \frac{1}{C}\int I_C(t)dt$$

This gives;

Eq24: $\frac{d}{dt} I_C(t) + \frac{I_C(t)}{RC} = 0$

We solve the differential equation:

$$\frac{d}{dt} I_C(t) = -\frac{I_C(t)}{RC}$$

Eq25: $\frac{d\, I_C(t)}{I_C(t)} = -\frac{1}{RC} dt$

Integrate both sides of Eq25:

$$\int \frac{d\, I_C(t)}{I_C(t)} = -\frac{1}{RC} \int dt$$

$$\ln I_C(t) + K1 = \frac{-t}{RC} + K2$$

Combine the constants of integration such that K3 = K2 – K1:

Eq26: $\ln I_C(t) = \frac{-t}{RC} + K3$

Solve Eq26 for $I_C(t)$:

$$e^{\left(\ln I_C(t)\right)} = e^{\left(\frac{-t}{RC} + K3\right)} = e^{\frac{-t}{RC}}\, e^{K3}$$

Let K4 = e^{K3}

Eq27: $I_C(t) = K4\, e^{\frac{-t}{RC}}$

We use the initial condition $I_C(0^+) = -\frac{E}{R}$ to give (Note: the minus sign denotes the fact that current is now flowing in the opposite direction):

Eq28: $K4 = -\frac{E}{R}$

The current $I_C(t)$ through capacitor C is given by:

Eq29: $I_C(t) = -\frac{E}{R}\, e^{\frac{-t}{RC}}$

- Capacitor voltage V_C (t)

The voltage across the resistor is given by:

$$V_R(t) = I_R(t)R = I_C(t)R = \left(-\frac{E}{R} e^{\frac{-t}{RC}} \right) R$$

Eq30: $V_R(t) = -Ee^{\frac{-t}{RC}}$

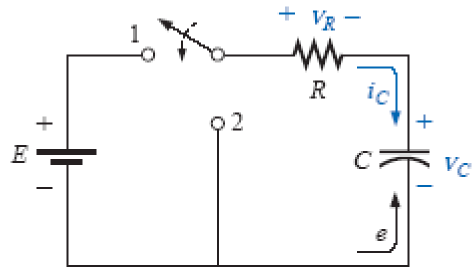
We know that:

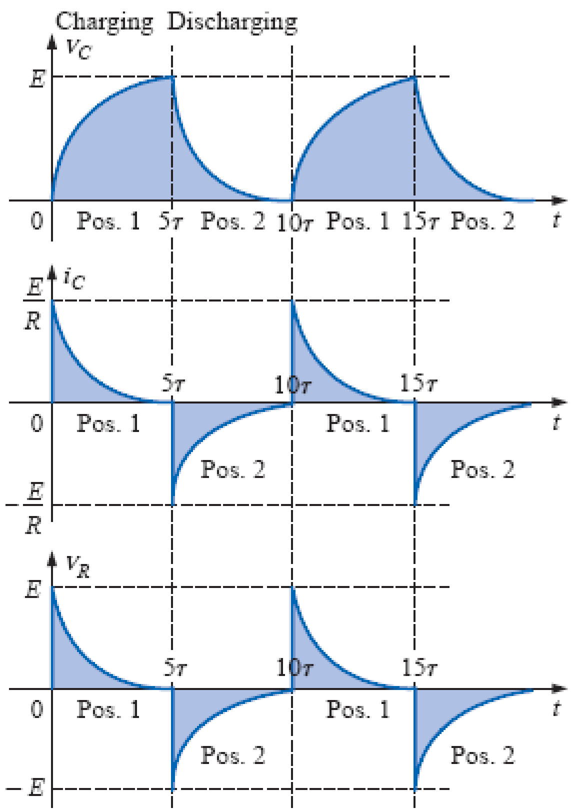
$$0 = V_R(t) + V_C(t)$$

$$V_C(t) = -V_R(t)$$

Eq31: $V_C(t) = Ee^{\frac{-t}{RC}}$

Summary – Capacitor Charging / Discharging phases

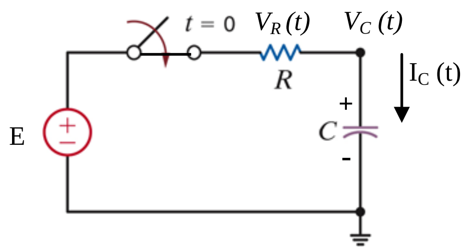




Assume $V_C(0^-) = 0$

Charging Phase	Discharging Phase
$V_C(t) = E\left(1 - e^{\frac{-t}{\tau}}\right)$	$V_C(t) = Ee^{\frac{-t}{RC}}$
$I_C(t) = \frac{E}{R}e^{\frac{-t}{\tau}}$	$I_C(t) = -\frac{E}{R}e^{\frac{-t}{RC}}$
$V_R(t) = E\,e^{\frac{-t}{RC}}$	$V_R(t) = -Ee^{\frac{-t}{RC}}$

1.3 Capacitor Transient Phases – Taking another look



@t = 0, the switch is closed providing a current path in the circuit. We recall Eq1, Eq2, and Eq4:

Eq1: $E = V_R(t) + V_C(t)$ → Eq1': $E = I_C(t)R + V_C(t)$

The voltage across a resistor and the current through a capacitor is defined as:

Eq2: $V_R(t) = I_C(t)R$

Eq4: $I_C(t) = C\frac{dV_C(t)}{dt}$

We solve for $V_C(t)$ thus, substitute Eq4 into Eq1' and divide by R on both sides

$$RC\frac{dV_C(t)}{dt} + V_C(t) = E$$

This first order linear differential equation with constant coefficients has a general solution comprising of two parts:

$$V_C(t) = V_{C(N)}(t) + V_{C(F)}(t)$$

- $V_{C(N)}$ is the complementary solution, (i.e. also called the natural response), whereby we set the right hand side of the differential equation to zero giving a homogeneous equation.
- $V_{C(F)}$ is the particular solution, (i.e. also called the forced response), whereby the solution is based on the particular right hand side of the equation.

We compute the Natural response $V_{C(N)}$:

$$\frac{dV_C(t)}{dt} + \frac{1}{RC}V_C(t) = 0$$

Let $V_{C(N)}(t) = Ke^{\alpha t}$

$$\frac{dKe^{\alpha t}}{dt} + \frac{1}{RC} Ke^{\alpha t} = 0$$

$$K\alpha e^{\alpha t} + \frac{1}{RC} Ke^{\alpha t} = 0$$

$$\left(\alpha + \frac{1}{RC}\right) Ke^{\alpha t} = 0$$

$$\alpha = -\frac{1}{RC}$$

Giving;

$$V_{C(N)}(t) = Ke^{-\frac{t}{RC}}$$

We compute the Forced response $\boldsymbol{V_{C\,(F)}}$:

Since the right hand side of the linear equation is a constant, then

$$V_{C(F)}(t) = A$$

$$RC\frac{dV_C(t)}{dt} + V_C(t) = E \rightarrow RC\frac{dA}{dt} + A = E$$

Giving;

A = E

The solution to the differential equation becomes:

$$V_C(t) = Ke^{\frac{-t}{RC}} + E$$

In order to solve for K, we use the initial condition $V_C\,(0) = 0$

$$Ke^0 + E = 0$$

$$K = -E$$

Substituting;

$$V_C(t) = -Ee^{\frac{-t}{RC}} + E$$

$$V_C(t) = E\left(1 - e^{\frac{-t}{\tau}}\right) \quad with\ \tau = RC$$

1.4 Capacitor Transient Phases – General Remarks

If voltages and currents in a 1st order RC circuit satisfy a differential equation of the form:

$$\frac{dx(t)}{dt} + a\,x(t) = f(t)$$

x(t) represents *i(t)* or *v(t)*

Where *f*(*t*) is the forcing function (i.e., the independent sources driving the circuit).

If *x(t)* = $V_C(t)$, then the solution of the differential equation at t > 0 takes the general form:

$$v_c(t) = K_1 + K_2 e^{\frac{-t}{\tau}} \quad \Rightarrow \quad \tau = R_{TH}\,C$$

R_{TH} is the circuit Thevenin resistance.
R_{TH} is found at t = ∞.

$$K_1 = v_C(\infty)$$

K_1 is referred as the steady state constant and is found when t = ∞

$$K_1 + K_2 = v_C(0^+)$$

K_2 is referred as the initial state constant and is found when t = 0^+

NOTE: In order to find $V_C(0^+)$, one needs to first find $V_C(0^-)$.
- $V_C(0^-) = V_C(\infty^-) \rightarrow V_C(0^+)$ as $V_C(t)$ cannot change instantaneously

$$K_2 = v_C(0^+) - v_C(\infty)$$

$$v_c(t) = v_C(\infty) + \left(v_C(0^+) - v_C(\infty)\right)e^{\frac{-t}{\tau}} \quad \Rightarrow \quad \tau = R_{TH}\,C$$

For the series RC circuit:

$K_1 = V_C(\infty) = E$

$E + K_2 = V_C(0^+) = 0 \rightarrow K_2 = -E$

Giving the same equation as found earlier:

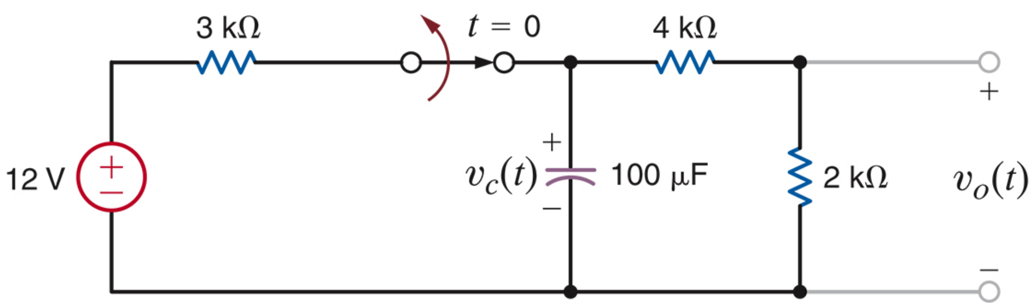
$$V_C(t) = E\left(1 - e^{\frac{-t}{\tau}}\right) \quad with\ \tau = RC$$

Example 1:

For the circuit below, @ t = 0, the switch opens. Find $v_o(t)$ @ t > 0

Let:

- R = 3 kΩ
- R_1 = 4 kΩ
- R_2 = 2 kΩ
- C = 100 uF



$$v_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, \quad t > 0$$

$$K_1 = v_C(\infty) = 0 \quad \text{(Capacitor fully discharged since switch is open)}$$

$$K_1 + K_2 = v_C(0+) = v_C(0-) = 12\left(\frac{4k\Omega + 2k\Omega}{3k\Omega + 4k\Omega + 2k\Omega}\right) = 8 \quad \text{(Switch closed)}$$

τ is the discharge time

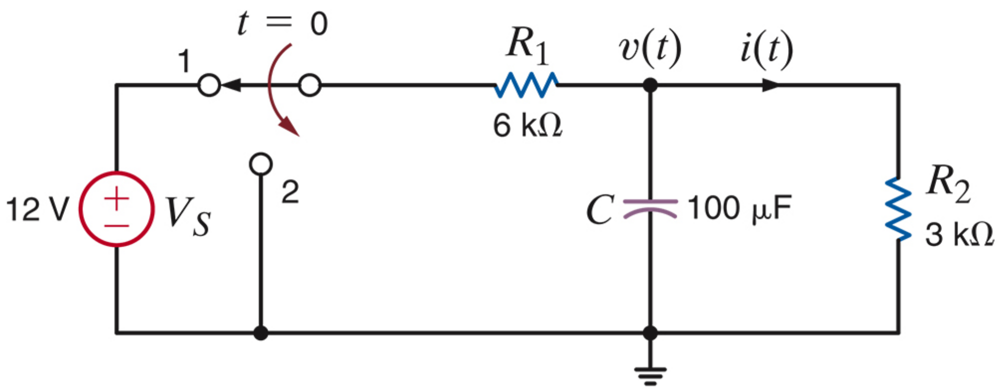
$$\tau = (R_1 + R_2)C = (6 \times 10^3 \Omega)(100 \times 10^{-6} F) = 0.6$$

$$v_C(t) = 8e^{-\frac{t}{0.6}}, \quad t > 0$$

$$v_O(t) = v_C(t)\frac{2k\Omega}{2k\Omega + 4k\Omega} = \frac{8}{3}e^{-\frac{t}{0.6}}, \quad t > 0$$

Example 2:

For the circuit below, @ t = 0, the switch moves to position 2. Find *i(t)* @ t > 0.



$$v_C(t) = K_1 + K_2 e^{-\frac{t}{\tau}}, \quad t > 0$$

$$i(t) = \frac{v_C(t)}{3k\Omega}$$

$$K_1 = v_C(\infty) = 0 \quad \text{(Capacitor fully discharged since switch is connected to position 2)}$$

$$K_1 + K_2 = v_C(0+) = v_C(0-) = 12\frac{3k}{3k+6k} = 4 \quad \text{(Switch at position 1)}$$

τ is the discharge time (Switch at position 2)

$$\tau = \frac{R_1 R_2}{R_1 + R_2} C = (2\times 10^3 \Omega)(100\times 10^{-6} F) = 0.2$$

$$v_C(t) = 4e^{-\frac{t}{0.2}}, \quad t > 0$$

$$i(t) = \frac{4}{3k\Omega} e^{-\frac{t}{0.2}}, \quad t > 0$$

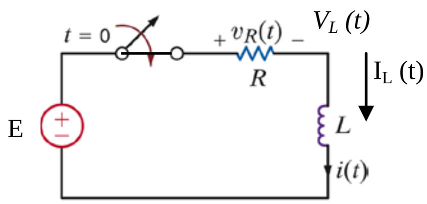
2 First Order RL Circuit Transient Analysis

Circuits containing only a single storage element are defined as first-order networks and result in a first-order differential equation (i.e. RC & RL circuits).

2.1 Inductor Storage Phase

- Inductor voltage V_L (t) with initial condition $I_L(0^-) = 0$

The RL Circuit provides a 1st order Differential Equation when performing the transient analysis:



Using Kirchhoff's voltage law:

Eq32: $E = V_R(t) + V_L(t)$

The voltage across a resistor, the current through an inductor and the voltage across an inductor is defined as:

Eq33: $V_R(t) = I_L(t)R$

Eq34: $I_L(t) = \frac{1}{L}\int_{t=-\infty}^{t} v(\tau)d\tau$

Eq35: $V_L(t) = L\frac{dI_L(t)}{dt}$

Substitute Eq33 and Eq34 into Eq32 and divide by R on both sides

Eq36: $E = \frac{R}{L}\int v_L(t)dt + v_L(t)$

Differentiate both sides of Eq36:

Eq37: $\frac{d}{dt}E = \frac{d}{dt}\left[\frac{R}{L}\int v_L(t)dt + v_L(t)\right]$

Rearrange and evaluate the derivatives of Eq37, giving a first order linear differential equation:

Eq38: $\frac{R}{L} v_L(t) + \frac{d}{dt} v_L(t) = 0$

We solve differential equation Eq38 by first rearranging the equation:

Eq39: $\frac{d\, v_L(t)}{v_L(t)} = -\frac{R}{L} dt$

Integrate both sides of Eq39:

$$\int \frac{d\, v_L(t)}{v_L(t)} = \int -\frac{R}{L} dt$$

$$\boldsymbol{ln}\, v_L(t) + K1 = \frac{-R}{L} t + K2$$

Combine the constants of integration such that K3 = K2 – K1:

Eq40: $\boldsymbol{ln}\, v_L(t) = \frac{-R}{L} t + K3$

Solve Eq40 for $v_L(t)$:

$$\boldsymbol{e}^{\left(\boldsymbol{ln}\, v_L(t)\right)} = e^{\left(\frac{-R}{L} t + K3\right)} = e^{\frac{-R}{L} t}\, e^{K3}$$

Let K4 = e^{K3}

Eq41: $v_L(t) = K4\, e^{\frac{-R}{L} t}$

We solve Eq41 for K4 using initial condition $v_L(0^+) = E$ since the model for an inductor at t = 0 is an open:

$$v_L(0^+) = E = K4\, e^0$$

Giving:

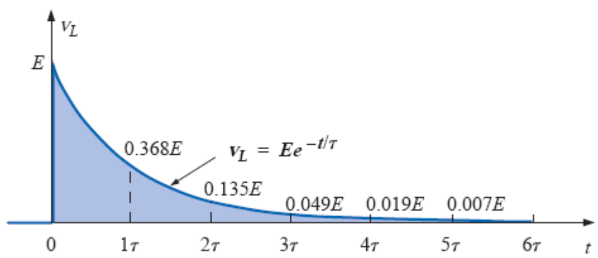
Eq42: $K4 = E$

Substitute Eq42 into Eq41 to get:

Eq43: $v_L(t) = E\, e^{\frac{-R}{L}t}$

We define L/R to be the network time constant with $\tau = L/R$ [sec].

We assume that steady state is reached at t = 5 τ



$(0 \rightarrow 1)\tau$	63.2%
$(1 \rightarrow 2)\tau$	23.3%
$(2 \rightarrow 3)\tau$	8.6%
$(3 \rightarrow 4)\tau$	3.0%
$(4 \rightarrow 5)\tau$	**1.2%**
$(5 \rightarrow 6)\tau$	0.4% ← Less than 1%

- Inductor current I_L (t) with initial condition $I_L(0^-) = 0$

We substitute Eq43 into Eq34 to derive the equation for $I_L(t)$ when $I_L(0^-) = 0$:

$$I_L(t) = \frac{1}{L}\int v_L(t)dt = \frac{1}{L}\int Ee^{\frac{-R}{L}t}dt = \frac{E}{L}\int e^{\frac{-R}{L}t}dt$$

We use integration by substitution with:

$$u = \frac{-R}{L}t\,,\ du = \frac{-R}{L}dt$$

Giving:

$$I_L(t) = -\frac{E}{R}\int e^{\frac{-R}{L}t}\frac{-R}{L}dt = -\frac{E}{R}\int e^{u}du = -\frac{E}{R}e^{u} + K$$

Eq44: $I_L(t) = -\frac{E}{R}e^{\frac{-R}{L}t} + K$

We solve Eq44 for K using the initial condition $I_L(0^-)=0$:

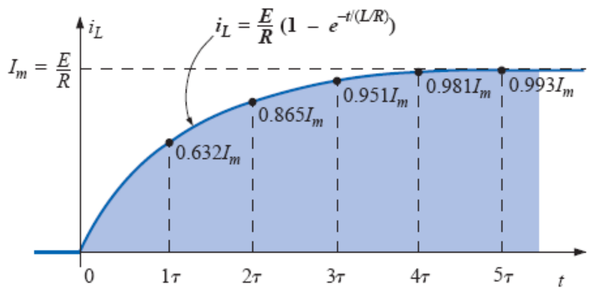
Eq45: $0=-\frac{E}{R}e^0+K \rightarrow K=\frac{E}{R}$

We substitute K into Eq44:

$$I_L(t)=\frac{E}{R}-\frac{E}{R}e^{\frac{-R}{L}t}$$

We can see that when $I_L(0^-)=0$, the voltage $I_L(t)$ through the inductor L is given by:

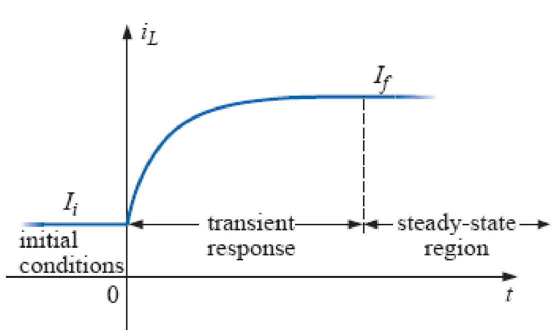
Eq46: $I_L(t)=\frac{E}{R}\left(1-e^{\frac{-R}{L}t}\right)=I_f\left(1-e^{\frac{-R}{L}t}\right)$ with $I_f=\frac{E}{R}$



- Inductor current I_L (t) with initial condition $I_L(0^-)=I_i$

With initial condition $I_L(0^-)=I_i$ the current through the inductor becomes:

Eq47: $I_L(t)=I_f-\left(I_f-I_i\right)e^{\frac{-R}{L}t}$



With initial condition $I_L(0^-) = I_i$ the voltage across the inductor becomes:

$$V_L(t) = (E - I_i R_W)e^{-tR/L}$$

For an ideal inductor, the winding resistance is negligible (i.e. 0Ω) and as such:

$$V_L(t) = Ee^{-tR/L}$$

The voltage across the resistor can be found by:

$$V_R(t) = E - V_L(t)$$

$$V_R(t) = E - Ee^{-\frac{R}{L}t}$$

Thus giving;

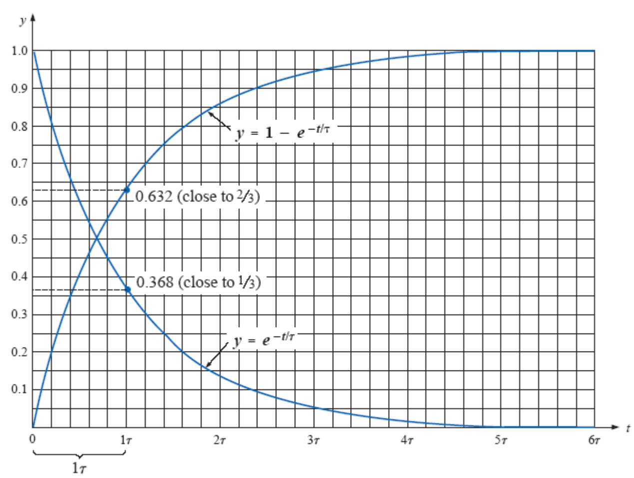
Eq48: $$V_R(t) = E\left(1 - e^{\frac{-R}{L}t}\right)$$

- Summary – Storage phase behavior for an RL circuit:

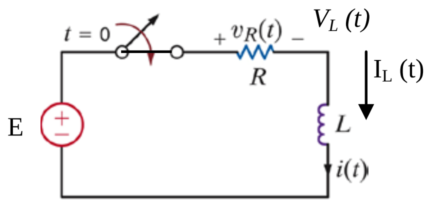
$$v_L(t) = E\, e^{\frac{-R}{L}t}$$

$$V_R(t) = E\left(1 - e^{\frac{-R}{L}t}\right)$$

$$I_L(t) = I_f - (I_f - I_i)e^{\frac{-R}{L}t}$$



2.2 Inductor Storing Phase – Taking another look



@t = 0, the switch is closed providing a current path in the circuit. We recall Eq32, Eq33, and Eq35:

Eq32: $E = V_R(t) + V_L(t)$ → Eq32': $E = I_L(t)R + V_L(t)$

The voltage across a resistor and the current through an inductor is defined as:

Eq33: $V_R(t) = I_L(t)R$

Eq35: $V_L(t) = L\dfrac{dI_L(t)}{dt}$

We solve for $I_L(t)$ thus, substitute Eq35 into Eq32' and divide by R on both sides

$$L\frac{dI_L(t)}{dt} + RI_L(t) = E$$

This first order linear differential equation with constant coefficients has a general solution comprising of two parts:

$$I_L(t) = I_{L(N)}(t) + I_{L(F)}(t)$$

- $I_{L\,(N)}$ is the complementary solution, (i.e. also called the natural response), whereby we set the right hand side of the differential equation to zero giving a homogeneous equation.

- $I_{L\,(F)}$ is the particular solution, (i.e. also called the forced response), whereby the solution is based on the particular right hand side of the equation.

We compute the Natural solution $I_{L\,(N)}$:

$$\frac{dI_L(t)}{dt} + \frac{R}{L}I_L(t) = 0$$

Let $I_{L(N)}(t) = Ke^{\alpha t}$

$$\frac{dKe^{\alpha t}}{dt} + \frac{R}{L}Ke^{\alpha t} = 0$$

$$K\alpha e^{\alpha t} + \frac{R}{L}Ke^{\alpha t} = 0$$

$$\left(\alpha + \frac{R}{L}\right)Ke^{\alpha t} = 0$$

$$\alpha = -\frac{R}{L}$$

Giving;

$$I_{L(N)}(t) = Ke^{-\frac{R}{L}t}$$

We compute the forced response $\boldsymbol{I_{L\,(F)}}$:

Since the right hand side of the linear equation is a constant, then

$$I_{L(F)}(t) = A$$

$$L\frac{dI_L(t)}{dt} + RI_L(t) = E \rightarrow L\frac{dA}{dt} + RA = E$$

Giving;

$$A = \frac{E}{R}$$

The solution to the differential equation becomes:

$$I_L(t) = Ke^{\frac{-R}{L}t} + \frac{E}{R}$$

In order to solve for K, we use the initial condition $I_L(0) = 0$

$$Ke^0 + \frac{E}{R} = 0$$

$$K = \frac{-E}{R}$$

Substituting;

$$I_L(t) = \frac{-E}{R}e^{\frac{-R}{L}t} + \frac{E}{R}$$

$$I_L(t) = \frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) \quad with\ \tau = \frac{L}{R}$$

2.3 Taking another look – General Remarks

If the voltages and currents in a 1st order RL circuit satisfy a differential equation of the form:

$$\frac{dx(t)}{dt} + a\,x(t) = f(t)$$

Where *f*(*t*) is the forcing function (i.e., the independent sources driving the circuit).

The solution of the differential equation at t > 0 takes the general form:

$$I_L(t) = K_1 + K_2 e^{\frac{-t}{\tau}} \quad \Rightarrow \quad \tau = \frac{L}{R_{TH}}$$

R_{TH} is the circuit Thevenin resistance

$$K_1 = I_L(\infty)$$

K_1 is referred as the steady state constant and is found when t = ∞

$$K_1 + K_2 = I_L(0^+)$$

K_2 is referred as the initial state constant and is found when $t = 0^+$

$$K_2 = I_L(0^+) - I_L(\infty)$$

$$I_L(t) = I_L(\infty) + \left(I_L(0^+) - I_L(\infty)\right)e^{\frac{-t}{\tau}} \quad \Rightarrow \quad \tau = \frac{L}{R_{TH}}$$

For the RL circuit:

$K_1 = I_L(\infty) = \frac{E}{R}$

$\frac{E}{R}$ + K2= IL (0+) = 0 → K2 = $-\frac{E}{R}$

Giving the same equation as found earlier:

$$I_L(t) = \frac{E}{R}\left(1 - e^{\frac{-t}{\tau}}\right) \quad with\ \tau = \frac{L}{R}$$